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Fractional branes on a non-compact orbifold

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Abstract: Fractional branes on the non-compact orbifold $\mathbb{C}^3/\mathbb{Z}_5$ are studied. First, the boundary state description of the fractional branes are obtained. The open-string Witten index calculated using these states reproduces the adjacency matrix of the quiver of $\mathbb{Z}_5$. Then, using the toric crepant resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_5$ and invoking the local mirror principle, B-type branes wrapped on the holomorphic cycles of the resolution are studied. The boundary states corresponding to the five fractional branes are identified as bound states of BPS D-branes wrapping the 0-, 2- and 4-cycles in the exceptional divisor of the resolution of $\mathbb{C}^3/\mathbb{Z}_5$.

Keywords: D-branes, Conformal Field Models in String Theory.

Contents

1. Introduction

D-branes wrapping supersymmetric cycles embedded in a variety purvey simple examples of curved D-branes. For special values of the moduli of string theory, as a cycle shrinks to null volume, leading, perhaps, to an orbifold singularity, the branes wrapped on it may give rise to states which are pinned to a point in the transverse space, perambulating only the Coulomb branch. Their charges and masses are certain fractions of that of a BPS D-brane; hence the neologism, *fractional brane* [1, 2]. Fractional branes on orbifolds appear also in the guise of boundary states of a perturbative superconformal field theory, graded by the irreducible representations of the quotienting group. The two descriptions are nonetheless related and pertain to different points in the moduli space of the conformal field theory (CFT). Confining our discussion to orbifolds $\mathbb{C}^3/G$ for some discrete subgroup G of $\mathrm{SL}(3,\mathbb{C})$, the wrapped branes furnish a valid description of these states at a point where the exceptional surfaces of the resolution of the orbifold are of large volume, while the CFT describes the states at the point where the exceptional surfaces shrink to vanishing volume giving rise to the orbifold singularity. We shall refer to the latter point as the *orbifold point* in the sequel. Discerning the guises of D-branes in different regions and exploring their interrelations are of import in obtaining geometric interpretations of the conformal field theoretic states, as well as clarifying several aspects of gauge field theories. Such studies have also provided physical realisations of the mathematical connection between representations of discrete groups and the cohomology of resolved orbifolds, namely, the McKay correspondence, in addition to providing evidence for mirror symmetry and its generalisations.

Different aspects of the identification of wrapped branes at the orbifold limit and the boundary states have been studied recently, from various points of view [3]–[16], confirming the geometric provenance of the fractional branes. Examples of threefolds on which the above scenario has been tested include D0-branes on blown up $\mathbb{C}^3/G$, for $G = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. The fractional D0-branes corresponding to $G = \mathbb{Z}_N$ have $1/N^{\mathrm{th}}$ the mass and charge of a D0-brane. In the large volume limit the fractional branes manifest themselves as bound states of branes wrapped on the holomorphic cycles of the resolution. For example, in the case of $\mathbb{C}^3/\mathbb{Z}_3$ [3, 5, 4], there are three fractional brane states, carrying one-third the mass and charge of a D0-brane. The exceptional surface of the resolution is a $\mathbb{P}^2$. In the large volume limit, BPS branes wrap on the holomorphic cycles of the $\mathbb{P}^2$. Thus, the fractional branes owe their origin to wrapped BPS D4-, D2- and D0-brane bound states. More precisely, they are identified with the trivial line bundle $\mathscr{O}$, the tautological line bundle $\mathscr{O}(-1)$ (up to signs) and an exceptional bundle of rank two on $\mathbb{P}^2$. The analysis has been generalised to other cases, especially to the partial resolutions of $\mathbb{C}^3/\mathbb{Z}_4$ and $\mathbb{C}^3/\mathbb{Z}_6$ [4]. These necessitated considering D4-branes wrapped on weighted projective spaces, $\mathbf{P}(\mathbb{C}^2)^{[112]}$ and $\mathbf{P}(\mathbb{C}^2)^{[123]}$ and hence bundles on these. The map between the fractional branes and the bundles on the divisors of the partial resolutions has been obtained in both these cases. Similar analyses on compact Calabi-Yau manifolds [8, 9, 10] yield a large volume D-brane interpretation of Gepner model states [17]–[20].

A feature common to the examples of threefolds mentioned above is that the Newton polygon corresponding to each of them is reflexive, i.e. contains a single point in its interior. In this article we extend certain aspects of such studies to another non-compact threefold, namely the crepant resolution $\mathbf{X}$ of $\mathbb{C}^3/G$, for $G = \mathbb{Z}_5 \subset \mathrm{SL}(3,\mathbb{C})$.

$$\begin{array}{ccc} & & \mathbf{X} \\ & & \downarrow \\ \mathbb{C}^3 & \xrightarrow{G} & \mathbb{C}^3/G \end{array} \tag{1.1}$$

The Newton polygon corresponding to the threefold $\mathbf{X}$ has two points in its interior. Thus, our result will furnish further evidence for the afore-mentioned scenario in a more general situation.

We shall consider D0-branes in type-IIA theory reduced on $\mathbb{C}^3/G$. We start by performing the quotienting by $G = \mathbb{Z}_5$ on the partition function of the $\mathcal{N} = 2$ theory. This allows us to identify the boundary states corresponding to the fractional branes. The identification of wrapped branes corresponding to these states in the resolution $\mathbf{X}$ closely parallels the example of $\mathbb{C}^3/\mathbb{Z}_3$ [3]. The five fractional branes on the orbifold form an orbit of the quantum $\mathbb{Z}_5$ symmetry. Since only the even-dimensional cohomologies of the resolution are non-trivial, the fractional D0-branes may occur only by wrapping of B-type branes in the Type-IIA theory on the holomorphic cycles in the large volume limit. We shall identify the set of wrapped branes which form an identical orbit of the monodromy around the orbifold point. In order to find the orbit in the large volume limit we shall obtain the volume of the cycles over the Kähler moduli space. This require determining the 0-, 2- and 4-cycle periods of $\mathbf{X}$, of which the 2- and 4-cycles have to be Legendre dual of each other with respect to the prepotential. The duals can not be identified correctly without the knowledge of the triple-intersection numbers of the divisors of $\mathbf{X}$. The triple-intersection numbers as calculated on $\mathbf{X}$ are known to be receive world-sheet instanton corrections. Considerations of world-sheet instanton corrections are avoided by taking recourse to the mirror geometry and hence working with the complex structure deformations of the mirror. For the non-compact threefold $\mathbf{X}$ at hand, the relevant version of mirror symmetry is the local one. We obtain the local mirror which is a Riemann surface of genus two for the present example. The large volume limit of the Kähler moduli space corresponds to the large complex structure limit (LCSL) in the mirror, while the orbifold limit is given by the vanishing of these deformations.

We then study the semi-periods [21] given by the $\mathcal{A}$-hypergeometric system of solutions to the associated GKZ system [22]–[26] in the LCSL, which is identified as the point of maximal degeneracy of the indicial equations of the GKZ system. The solutions are then continued analytically to the orbifold point and we find out the monodromy of periods around that point. There are five charged states in this regime which form periods of the monodromy matrix. By virtue of being of equal mass, these can be identified with the fractional branes. The BPS-brane content of these states can be read off from the charges of 0-, 2- and 4-cycle periods.

The plan of the paper is as follows. First, we obtain the boundary states for the fractional branes on the orbifold in section 2. We then describe the toric resolution of the orbifold, its local mirror geometry and describe the GKZ system for the mirror in section 3. The periods in the LCSL are obtained by solving the GKZ system in section 4. Finally, in section 5, we analytically continue the periods to the orbifold point and find out the monodromies around the orbifold point which is then used for the identification of the fractional branes with the wrapped branes at large volume. We summarise the results in section 6.

2. Open strings on the orbifold $\mathbb{C}^3/G$

The orbifold $\mathbb{C}^3/\mathbb{Z}_5$ is defined by the action of $\mathbb{Z}_5$ on $\mathbb{C}^3$ through the defining representation $\operatorname{diag}\omega, \omega, \omega^3$ on the three complex coordinates,

$$\mathbb{Z}_5 : (z_1, z_2, z_3) \longmapsto (\omega^{a_1} z_1, \omega^{a_2} z_2, \omega^{a_3} z_3), \quad (z_1, z_2, z_3) \in \mathbb{C}^3 , \tag{2.1}$$

where ω is a fifth root of unity: $\omega = e^{\frac{2\pi i}{5}}$, $a_1 = a_2 = 1$ and $a_3 = 3$. The orbifold is also denoted $\frac{1}{5}[113]$. In this section we shall first consider the string partition function in the open string channel. The action of the group G is effected on the partition function. Then using the modular properties of theta functions, the same is translated to expressions in terms of closed string variables and the partition function is expressed as a transition amplitude between boundary states. Finally, we calculate the open string Witten index as the transition amplitude between the I^{th} and the J^{th} states in the R-sector. Collecting them in a matrix produce the adjacency matrix of the Quiver diagram of G.

NB: Throughout this section alone $\prod$ abbreviates $\displaystyle\prod_{n=1}^{\infty}$.

2.1 Partition function in the open string channel

D-branes carry the Chan-Paton charges of open strings. Thus, in addition to specifying the action of G on $\mathbb{C}^3$, we have to specify its action on the Chan-Paton degrees of freedom. The action of G on the Chan-Paton indices, leading to one single D0-brane [27, 28] is consistent within string theory only if G acts on the Chan-Paton matrices through its regular representation, namely $\mathcal{R} = \bigoplus_I d_I R_I$, where R_I denotes the I^{th} irreducible representation, $I = 1, \ldots, |G|$ and $|G|$ denotes the order of the group. Thus, for the case at hand, $|G| = 5$. The multiplicities $d_I = 1$ for all I. However, we shall keep the d_I for book-keeping; the formulas later in this section are actually valid in more general situations. Furthermore, since G is cyclic, the elements of G can be written as g^m, $m = 0, \ldots, |G| - 1$, for an element g of G.

We consider the open string connecting I^{th} and J^{th} representations. The G-projected partition function in a sector $\mathfrak{s}$ can be written as

$$\mathcal{Z}_{\mathfrak{s}} = \frac{d_I d_J}{|G|} \sum_{m=0}^{|G|-1} \bar{\chi}^I(m) \chi^J(m) \mathrm{Tr}_{\mathfrak{s}} \left(g^m e^{-2\tau H_o} \right) , \tag{2.2}$$

where $\mathfrak{s}$ is either the NS or the R sector, in which the trace is taken and $\chi^I(m)$ denotes the I^{th} character of g^m. Moreover, H_o denotes the open string Hamiltonian and is given by

$$H_o = \pi p^2 + \pi \sum_{\mu=0,\ldots,7} \left(\sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_n^{\mu} + \sum_{r>0} r \psi_{-r}^{\mu} \psi_r^{\mu} \right) + \pi c_0 \,, \tag{2.3}$$

where α refers to the oscillators, p denotes the momentum and c_0 is a constant [3].

We now implement the action of G on the different sectors of the partition function. The partition function in the NS-sector is

$$\begin{aligned} \mathcal{Z}_{\mathrm{NS}}(0,\tau) &= \frac{1}{\eta(\tau)^8} \left[\frac{\vartheta_3(\tau)}{\eta(\tau)} \right]^4 \\ &= q^{-1/2} \frac{\prod (1+q^{n-1/2})^8}{\prod (1-q^n)^8} \,, \end{aligned} \tag{2.4}$$

where $q = e^{2i\pi\tau}$. Reduced on $\mathbb{C}^3$ and projected by g^m, it takes the form

$$\begin{aligned} \mathcal{Z}_{\mathrm{NS}}(m,\tau) &= q^{-1/2} \left[\frac{\prod (1+q^{n-1/2})}{\prod (1-q^n)} \right]^2 \prod_{i=1}^{3} \frac{\prod (1+\omega^{a_i} q^{n-1/2})(1+\bar{\omega}^{a_i} q^{n-1/2})}{\prod (1-\omega^{a_i} q^n)(1-\bar{\omega}^{a_i} q^n)} \,, \\ &= 8 \, \frac{\vartheta_3(\tau)}{\eta(\tau)^3} \prod_{i=1}^{3} \frac{\vartheta_3(\frac{ma_i}{N}|\tau)}{\vartheta_1(\frac{ma_i}{N}|\tau)} \sin\left(\frac{\pi m a_i}{N} \right) . \end{aligned} \tag{2.5}$$

Similarly, from the unprojected partition function of the $(-1)^F$NS-sector, *viz.*

$$\mathcal{Z}_{(-1)^F \mathrm{NS}}(0) = \frac{1}{\eta(\tau)^8} \left[\frac{\vartheta_4(\tau)}{\eta(\tau)} \right]^4 \,, \tag{2.6}$$

we derive the following after projection by g^m

$$\mathcal{Z}_{(-1)^F \mathrm{NS}}(m,\tau) = 8 \, \frac{\vartheta_4(\tau)}{\eta(\tau)^3} \prod_{i=1}^{3} \frac{\vartheta_4(\frac{ma_i}{N}|\tau)}{\vartheta_1(\frac{ma_i}{N}|\tau)} \sin\left(\frac{\pi m a_i}{N} \right) . \tag{2.7}$$

The partition function in the R-sector, on the other hand, after projection, takes the form

$$\mathcal{Z}_{\mathrm{R}}(m,\tau) = 8 \, \frac{\vartheta_2(\tau)}{\eta(\tau)^3} \prod_{i=1}^{3} \frac{\vartheta_2(\frac{ma_i}{N}|\tau)}{\vartheta_1(\frac{ma_i}{N}|\tau)} \sin\left(\frac{\pi m a_i}{N} \right) . \tag{2.8}$$

Collecting contributions from the different sectors after projection, the partition function on $\mathbb{C}^3/G$ reads

$$\tilde{\mathcal{Z}}(m,\tau) = \mathcal{Z}_{\mathrm{NS}}(m,\tau) - \mathcal{Z}_{(-1)^F \mathrm{NS}}(m,\tau) - \mathcal{Z}_{\mathrm{R}}(m,\tau). \tag{2.9}$$

The total partition function is obtained by summing this over all the elements of the group,

$$\mathcal{Z} = \frac{d_I d_J}{|G|} \sum_{m=0}^{N-1} \hat{\mathcal{Z}}(m,\tau) \, \bar{\chi}^I(m) \chi^J(m) \,, \tag{2.10}$$

where

$$\hat{\mathcal{Z}}(m,\tau) = \frac{V}{2\pi} \int \frac{dt}{4\sqrt{2} t^{3/2}} \tilde{\mathcal{Z}}(m,\tau) \tag{2.11}$$

and $t = -i\tau$.

2.2 Boundary states

Boundary states provide a closed string theoretic description of the CFT and can be obtained from the partition function in the open string sector [3, 29, 30]. The basic idea is to find the Ishibashi states in the closed string CFT, which satisfy certain linearized boundary conditions. However all such states do not correspond to a state describing a D-brane. In order to qualify as a physical state, they have to satisfy the Cardy's conditions that the closed string scattering amplitude should factorise to appropriate sectors in the open string channel. Moreover, the physical states will have to survive the GSO projection. In the case of orbifold, due to conformal invariance we have an additional factorization condition on the boundary states [3].

In order to identify the boundary state, let us write the open string partition function (2.10) in terms of closed string variables as

$$\mathcal{Z} = \frac{d_I d_J}{|G|} \sum_{m=0}^{N-1} \hat{\mathcal{Z}}_{\rm cl}(m,l)\, \bar{\chi}^I(m) \chi^J(m)\,, \tag{2.12}$$

where

$$\hat{\mathcal{Z}}_{\rm cl}(m,l) = \frac{iV}{2\pi} \int \frac{dl}{l^{3/2}} \tilde{\mathcal{Z}}_{\rm cl}(m,l)\,, \tag{2.13}$$

and

$$\begin{aligned}
\tilde{\mathcal{Z}}_{\rm cl}(m,l) \;=\; & \frac{\vartheta_3(2il)}{\eta(2il)^3} \prod_{i=1}^{3} \frac{\vartheta_3(\frac{-2ma_i l}{N}|2il)}{\vartheta_1(\frac{-2ma_i l}{N}|2il)} \sin\left(\frac{\pi m a_i}{N}\right) - \\
& - \frac{\vartheta_4(2il)}{\eta(2il)^3} \prod_{i=1}^{3} \frac{\vartheta_4(\frac{-2ma_i l}{N}|2il)}{\vartheta_1(\frac{-2ma_i l}{N}|2il)} \sin\left(\frac{\pi m a_i}{N}\right) - \\
& - \frac{\vartheta_2(2il)}{\eta(2il)^3} \prod_{i=1}^{3} \frac{\vartheta_2(\frac{-2ma_i l}{N}|2il)}{\vartheta_1(\frac{-2ma_i l}{N}|2il)} \sin\left(\frac{\pi m a_i}{N}\right),
\end{aligned} \tag{2.14}$$

using modular transformation properties of the theta functions [3].

In the closed string channel we have the Ishibashi states satisfying appropriate Neumann-Dirichlet conditions with the different spin structures. They are the coherent states obtained by acting exponentiated sum of oscillators on the vacuum. For every twisted sector there is an Ishibashi state with a given spin structure from NS-NS and R-R sectors. Boundary states are obtained by writing (2.12) as transition amplitudes between such states. This is achieved using the expressions of the transition amplitudes between Ishibashi states in terms of theta functions in each twisted sector in the closed string channel [3],

$$\begin{aligned}
\int dl \langle +, m|\, e^{-lH_{\rm cl}} |+, m\rangle_{\rm NS-NS} &= \int dl \langle -, m|\, e^{-lH_{\rm cl}} |-, m\rangle_{\rm NS-NS} \\
&= i\mathsf{N}_m^2 \int \frac{dl}{l^{3/2}}\, \vartheta_3(2il) \prod_{i=1}^{3} \frac{\vartheta_3(\frac{-2ma_i l}{N}|2il)}{\vartheta_1(\frac{-2ma_i l}{N}|2il)}\,,
\end{aligned} \tag{2.15}$$

$$
\begin{aligned}
\int dl \langle +, m|\, e^{-lH_{\rm cl}} |-, m\rangle_{\rm NS-NS} &= \int dl \langle -, m|\, e^{-lH_{\rm cl}} |+, m\rangle_{\rm NS-NS} \\
&= i\mathsf{N}_m^2 \int \frac{dl}{l^{3/2}}\, \vartheta_2(2il) \prod_{i=1}^{3} \frac{\vartheta_2(\frac{-2ma_i l}{N}|2il)}{\vartheta_1(\frac{-2ma_i l}{N}|2il)}\,, \\
\int dl \langle +, m|\, e^{-lH_{\rm cl}} |+, m\rangle_{\rm RR} &= \int dl \langle -, m|\, e^{-lH_{\rm cl}} |-, m\rangle_{\rm RR} \\
&= -i\mathsf{N}_m^2 \int \frac{dl}{l^{3/2}}\, \vartheta_4(2il) \prod_{i=1}^{3} \frac{\vartheta_4(\frac{-2ma_i l}{N}|2il)}{\vartheta_1(\frac{-2ma_i l}{N}|2il)}\,, \qquad (2.16)
\end{aligned}
$$

with the normalisation determined by demanding equality of the expressions in the two channels as

$$
\mathsf{N}_m^2 = \frac{d_I^2}{|G|} \prod_{i=1}^{3} \sin\left(\frac{\pi m a_i}{N}\right) |\chi^I(m)|^2\,, \qquad (2.17)
$$

from which we can find out the normalizations of the Ishibashi states with different spin structures associated to a representation I. In these formulas $H_{\rm cl}$ denoting the closed string Hamiltonian.

Boundary states are then obtained by comparing the different terms in the partition function (2.12) and the above amplitudes, both now in the closed string variables.

$$
|\mathcal{B}, m, I\rangle = \frac{1}{2}\left(|+, m, I\rangle_{\rm NS-NS}| - |-, m, I\rangle_{\rm NS-NS} + |+, m, |I\rangle_{\rm R-R} + |-, m, |I\rangle_{\rm R-R}\right), \qquad (2.18)
$$

for $m = 0, \ldots, N-1$. The index m labels the state that pertains to the element g^m of G and I the irreducible representation. The Chan-Paton indices are collected in $\mathcal{B}$. The signs ($\pm$) stand for the respective spin structures. We now write the states in a basis corresponding to the nodes of the quiver diagram, which, in turn, corresponds to the irreducible representations as [30]

$$
|\,\mathcal{B}, I\rangle = \sum_{m=0}^{|G|-1} |\,\mathcal{B}, m\rangle\,. \qquad (2.19)
$$

Let us point out that the boundary states (2.19) are graded by the characters of the representation of G through I. This one-to-one correspondence between the boundary states and the characters of representation is the crucial ingredient in the identification of these boundary states as branes wrapping homology cycles in the resolution of the orbifold, which may be thought of as a physical realisation of the McKay correspondence [31].

2.3 The open-string Witten Index

Let us now consider the partition function in the RR-sector in a little more detail.

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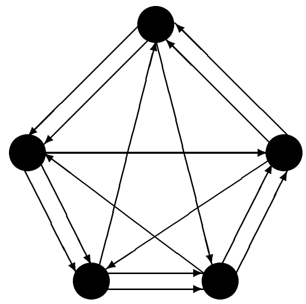

Figure 1: Quiver diagram for $G = \mathbb{Z}_5$.

Extracting the contribution from the RR-sector in $\hat{\mathcal{Z}}_{\text{cl}}$, we have

$$\mathcal{Z}_{\text{R}}(\tau) = \frac{8 d_I d_J}{|G|} \frac{\vartheta_2(\tau)}{\eta(\tau)^3} \sum_{m=0}^{N-1} \prod_{i=1}^{3} \frac{\vartheta_2(\frac{m a_i}{N}|\tau)}{\vartheta_1(\frac{m a_i}{N}|\tau)} \sin\left(\frac{\pi m a_i}{N}\right). \tag{2.20}$$

Accordingly, the contribution to $\mathcal{Z}$ of (2.12) from the massless part of $\mathcal{Z}_{\text{R}}(\tau)$, after GSO projection, written as a matrix in the I indices, yields the matrix of Witten indices [29, 30],

$$\mathcal{W}_{IJ} = \frac{8 d_I d_J}{|G|} \sum_{m=0}^{N-1} \prod_{i=1}^{3} (-1)^m \cos\left(\frac{\pi m a_i}{N}\right) \bar{\chi}^I(m) \chi^J(m). \tag{2.21}$$

Using $d_I = 1$ for all I for the regular representation and the character table for $G = \mathbb{Z}_5$, shown in table 1, we obtain the matrix $\mathcal{W}$ for the present example.

The matrix $\mathcal{W}$ reproduces the adjacency matrix of the quiver diagram of $G = \mathbb{Z}_5$, *viz.*

$$\mathcal{C} = \begin{pmatrix} 0 & 2 & -1 & 1 & -2 \\ -2 & 0 & 2 & -1 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ -1 & 1 & -2 & 0 & 2 \\ 2 & -1 & 1 & -2 & 0 \end{pmatrix}, \tag{2.22}$$

up to signs. Thus, $\mathcal{W} = |\mathcal{C}|$. Let us emphasise that the boundary states that yield the adjacency matrix are on the *orbifold* $\mathbb{C}^3/\mathbb{Z}_5$. Later, we shall find that the same adjacency matrix of the quiver is reproduced by the intersection pairing of the bound states of wrapped branes. This information along with the grading of the boundary states by the characters lead to the identification of the boundary states as bound states of wrapped branes on the resolution of the orbifold.

I^m	0	1	2	3	4
0	1	1	1	1	1
1	1	ω	ω^2	ω^3	ω^4
2	1	ω^2	ω^4	ω	ω^3
3	1	ω^3	ω	ω^4	ω^2
4	1	ω^4	ω^3	ω^2	ω

Table 1: Characters $(\chi^I(m))$ of $G = \mathbb{Z}_5$.

3. D-branes on the blow up

In the previous section we have obtained the boundary states in the CFT describing fractional branes on the orbifold $\mathbb{C}^3/\mathbb{Z}_5$. In order to interpret these states geometrically, we have to describe BPS branes wrapped on the exceptional divisors of the resolution $\mathbf{X}$ in the large volume limit and show the emergence of the fractional branes as the cycles shrink at the orbifold. In other words, we need to find the behaviour of the BPS-spectrum of the orbifold theory under the marginal deformations which parametrise the Kähler moduli space of $\mathbf{X}$. This, added to the fact that the boundary states are in one-to-one correspondence with the irreducible representations of $G = \mathbb{Z}_5$, furnish a physical realisation of the McKay correspondence [31]–[34]. However, as mentioned in section 1, in order to avoid dealing with world-sheet instanton corrections to the Kähler structure moduli space, we shall go over to the complex structure moduli space of the mirror geometry. The latter follows from the periods of a meromorphic one-form on the mirror Riemann surface. The periods solve a set of differential equation known as the GKZ system of equations or the generalised hypergeometric system. The solutions to the GKZ system are also known as semi-periods. We shall use the expressions semi-period and period interchangeably in this article.

In order to set up the notations let us recall a few notions in toric geometry which are relevant for our purpose [35]–[39]. An n-dimensional toric variety $\mathcal{V}$ is represented combinatorially as a fan Σ in a real vector space $\mathbf{N}_{\mathbb{R}}$ associated with a lattice $\mathbf{N}$ which consists of a set of strongly convex rational polyhedral cones σ. A fan is described by a set of k points $(k > n)$ $\{\vec{v}_i\}$ (which we shall refer to as the toric data) in $\mathbf{N}$ and the cones are given upon its triangulation. The convex hull of these points is called the Newton polytope $\triangle_N$. The points satisfy a set of $(k-n)$ linear relations $\{\ell_i^a\}$, *e.g.* $\sum_i l_i^a \vec{v}_i = 0$. Each of the points $\vec{v}_i$ corresponds to a homogeneous variable x_i of $\mathbb{C}^k$, while a relation specifies a $\mathbb{C}^\star$ action on the homogeneous coordinates. This yields the the n-dimensional toric variety as a quotient $\mathcal{V} = (\mathbb{C}^k \setminus U)/(C^\star)^{k-n}$. U is the set of singular points of the $\mathbb{C}^\star$ actions which is determined by the triangulation of the fan.

To every fan in the lattice $\mathbf{N}_{\mathbb{R}}$ is associated a dual fan consisting of cones in the dual space $\mathbf{M}_{\mathbb{R}}$ where $\mathbf{M}$ =Hom$(\mathbf{N}, \mathbb{Z})$ is the dual lattice. The cone $\overset{\vee}{\sigma}$ dual to σ is given by $\overset{\vee}{\sigma}= \{u \in \mathbf{M}_{\mathbb{R}} | \langle u, v\rangle \geq 0, \forall v \in \sigma\}$. The dual of the Newton polytope in $\mathbf{M}_{\mathbb{R}}$ is given as $\nabla_N = \{u \in \mathbf{M}_{\mathbb{R}} | \ \langle u, v\rangle \geq -1, \forall v \in \triangle_N\}$. The toric data associated with the dual fan are the points representing the vertices of this dual polytope. The dual polytope defines a variety dual to $\mathcal{V}$ similarly as above and for a large variety of cases these two varieties are related to each other by mirror symmetry.

In a fan associated to an orbifold, the singularity is signalled by the presence of a cone which does not have unit volume in units of lattice-spacing. A singularity can be resolved by refining the triangulation of the fan.

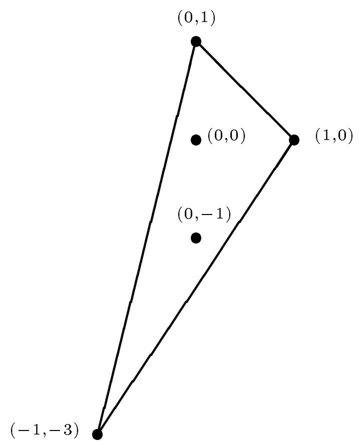

Figure 2: Plot of the columns of the toric data $\mathcal{A}$ on the height-one plane — the Newton polygon Δ.

In this section, we shall first briefly describe the toric resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_5$, followed by the triple-intersection numbers of the exceptional divisors. We then discuss the local mirror geometry, which is a Riemann surface of genus two, given as a polynomial equation of degree 5. The GKZ equations for the mirror geometry are then derived from the toric data. Finally, we obtain the semi-periods as solutions of the GKZ system in the LCSL.

3.1 Resolution of $\mathbb{C}^3/\mathbb{Z}_5$ by D-branes

A D0-brane on the blown up orbifold $\mathbb{C}^3/\mathbb{Z}_5$ is obtained by starting with the supersymmetric linear sigma model describing the world-volume gauge theory of five coalesced D0-branes and then identifying these under the $\mathbb{Z}_5$ symmetry. This is effected by an action of $\mathbb{Z}_5$ on the Chan-Paton matrices of the open strings ending on the branes by the regular representation of $\mathbb{Z}_5$. The vacuum moduli space of the world-volume gauge theory, obtained by solving the F- and D-flatness conditions, provides the resolution $\mathbf{X}$ of $\mathbb{C}^3/\mathbb{Z}_5$. The resolution $\mathbf{X}$ is described as a toric variety by the charges of the chiral multiplets in the linear sigma model [28], and is given as the following toric data

$$\begin{array}{c} \begin{array}{ccccc} \mathfrak{D}_0 & \mathfrak{D}_3 & \mathfrak{D}_4 & \mathfrak{D}_1 & \mathfrak{D}_2 \end{array} \\ \mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \end{array} \tag{3.1}$$

The five columns of the matrix $\mathcal{A}$ define a set of five integral points on the height-one plane of the third coordinate in the lattice $\mathbf{N} \cong \mathbb{Z}^3 \subset \mathbb{R}^3$. The convex hull of these points in $\mathbb{R}^3$ is the Newton polytope. Its trace on the height-one plane is the Newton polygon Δ, as shown in figure 2. We have denoted the toric divisors corresponding to the points in the Newton polygon by $\mathfrak{D}$. From the toric data $\mathcal{A}$ we see that the toric divisors are linearly related to each other,

$$\begin{aligned}
\mathfrak{D}_2 &= \mathfrak{D}_3 \stackrel{\text{def}}{=} \mathfrak{d}_1 \\
\mathfrak{D}_4 &\stackrel{\text{def}}{=} \mathfrak{d}_2 \\
\mathfrak{D}_0 &= \mathfrak{d}_1 - 2\mathfrak{d}_2 \\
\mathfrak{D}_1 &= \mathfrak{d}_2 - 3\mathfrak{d}_1 \,,
\end{aligned} \tag{3.2}$$

where we introduced the two linearly independent divisors $\mathfrak{d}_1$ and $\mathfrak{d}_2$, which can be taken to be the generators of the Picard group, $\operatorname{Pic}\mathbf{X} = \mathbb{Z} \oplus \mathbb{Z}$.

The resolution $\mathbf{X}$ of the orbifold corresponds to triangulating on all the points of the Newton polygon, as shown in figure 3. Let us point out that the same triangulation of the Newton polygon can also be achieved, playfully, by observing the resemblance between the triangulation in figure 3 and the junior simplex [32, 31] corresponding to the orbifold, by the survivours of the *champions' meet* [33, 34], as in figure 4. The Newton polygon is not reflexive, as it has two points in its interior implying $h^{11} = 2$ for the resolved orbifold. The divisors $\mathfrak{D}_0$ and $\mathfrak{D}_1$, corresponding to the internal points are the exceptional divisors. As shown in figure 3 and figure 4, the two interior points corresponding to $\mathfrak{D}_0$ and $\mathfrak{D}_1$ have line-valency 4 and 3, corresponding, thereby, to a rational scroll $\mathbb{F}_2$, which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and the projective plane $\mathbb{P}^2$, respectively [31, 33, 34].

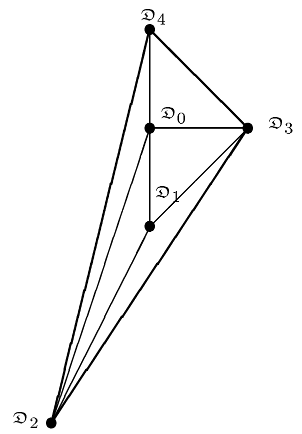

Figure 3: Triangulation of the Newton polygon.

The Gale diagram of the 3×4 matrix $\mathcal{A}$, assuming value in the kernel of $\mathcal{A}$, is

$$\mathcal{G} = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \\ -3 & 1 \\ 1 & 0 \end{pmatrix} \in \operatorname{Ker}\mathcal{A}\,. \tag{3.3}$$

To the five three-vectors of the Newton polyhedron is associated a lattice of linear relations, generated by the columns of the Gale diagram. Let us denote it by $\mathcal{L}_{\mathcal{A}}$; it is a sublattice of $\mathbb{Z}^5$. The canonical bundle of $\mathbf{X}$ is trivial, as may be seen from the

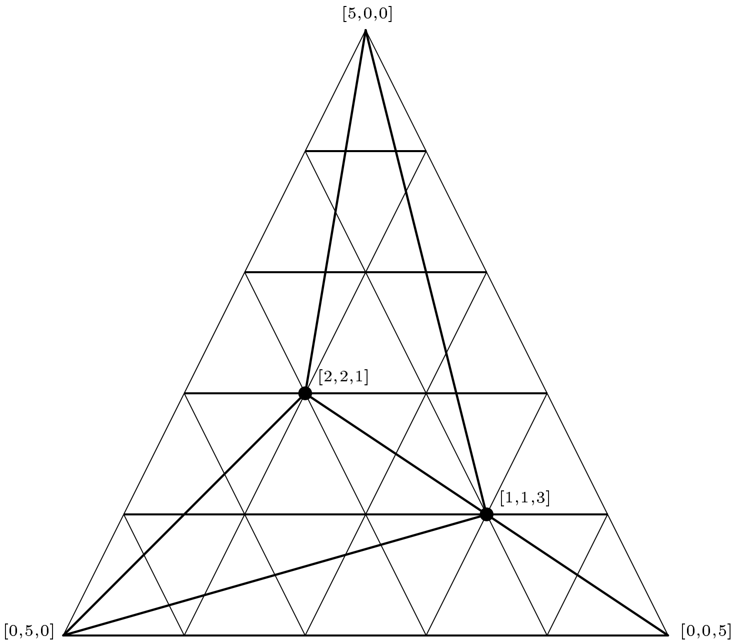

Figure 4: The junior simplex for $\frac{1}{5}[113]$.

fact that the sum of the components of each element ℓ of $\mathcal{L}_{\mathcal{A}}$ vanishes,

$$\sum_{i=0}^{4} \ell_i = 0\,, \quad \ell \in \mathcal{L}_{\mathcal{A}}\,, \tag{3.4}$$

signifying that the resolution is crepant. From the relations

$$\begin{aligned} \ell^{(1)} &= (\,1 \quad 1 \quad 0 \quad -3 \quad 1\,) \\ \ell^{(2)} &= (\,\text{-2} \quad 0 \quad 1 \quad 1 \quad 0\,) \end{aligned} \tag{3.5}$$

we can write down the toric ideal of $\mathbf{X}$, namely, $\langle y_0y_1y_4 - y_3^3,\, y_2y_3 - y_0^2\rangle$, where we have written the generators of the ideal in Gröbner basis. Consequently, the relations ℓ^I generate the Mori cone of the variety associated to Δ.

3.2 Yukawa couplings

In order to study the B-type branes wrapped on the cycles corresponding to the meromorphic form on $\overset{\vee}{\mathbf{X}}$, we need to derive the periods as solutions to the GKZ system. To write the solutions in the LCSL in the canonical basis we need to know the triple-intersection numbers of the divisors, alias Yukawa couplings. Let us consider the triangulation of the Newton polygon shown in Figure 3. Denoting the intersection number $\mathfrak{D}_i \cap \mathfrak{D}_j \cap \mathfrak{D}_k$ by C_{ijk}, we have the following values for Yukawa couplings [41, 42], corresponding to the two divisors $\mathfrak{D}_0$ and $\mathfrak{D}_1$,

$$\begin{aligned} C_{000} &= 8 \qquad & C_{001} &= 1 \\ C_{011} &= -3 \qquad & C_{111} &= 9\,, \end{aligned} \tag{3.6}$$

the other couplings vanish. Using the relations (3.2) we can derive the triple intersection numbers corresponding to the divisors $\mathfrak{d}$,

$$\begin{aligned} c_{111} &= -\frac{2}{5} \qquad & c_{112} &= -\frac{1}{5} \\ c_{122} &= -\frac{3}{5} \qquad & c_{222} &= -\frac{9}{5}\,, \end{aligned} \tag{3.7}$$

where now c_{ijk} denotes the intersection number of $\mathfrak{d}_i \cap \mathfrak{d}_j \cap \mathfrak{d}_k$. In the next section we shall use the couplings c_{ijk} to obtain the solutions to the GKZ system (3.19) in the LCSL.

3.3 The mirror

As mentioned earlier, in order to obtain the exact Kähler moduli space of the orbifold we need to consider the moduli space of complex structure of the mirror geometry. The construction of local mirror $\overset{\vee}{\mathbf{X}}$ of $\mathbf{X}$ follows by making use of the fact that a pair of dual Newton polygons in dual lattice spaces N and M defines a pair of mirror varieties. We shall write the mirror of the non-compact variety as a curve [36]–[40] defined by the polynomial equation

$$\sum_{i=0}^{4} b_i y_i = 0\,, \tag{3.8}$$

where the y_i appeared in the binomials of the generators of the toric ideal and the b_i are five complex numbers. The coefficients of the monomials represent the deformation of complex structure or equivalently the blow up of the Kähler structure of the mirror geometry. However, all of them are not independent and the appropriate choice is associated with the choice of the kernel. In order to make use of the mirror map one has to choose the Mori basis of kernel so that the independent coordinates of complex structure moduli space of mirror geometry correspond to the generators of the Kähler cone in the moduli space of Kähler structure in the blown up geometry. The moduli space is singular in general. The singular set is given by the principal component of the discriminant.

We can rewrite the equation in terms of homogeneous variables $\tilde{y}_i$ as

$$P(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \equiv b_1\tilde{y}_1^5 + b_4\tilde{y}_2^5 + b_2\tilde{y}_3^5 + b_0\tilde{y}_1\tilde{y}_2\tilde{y}_3^3 + b_3\tilde{y}_1^2\tilde{y}_2^2\tilde{y}_3 = 0\,. \tag{3.9}$$

Let us note at this point the usefulness of the junior simplex in obtaining the homogeneous variables. The indices of the homogeneous variables in the monomials in the above equation are given by the points marked in figure 4 that participate in the triangulation. We then rescale the coordinates $\tilde{y}_i$, using the $(\mathbb{C}^\star)^3$ symmetry

$$\begin{aligned} \tilde{y}_1 &= x_1 b_1^{-1/5}\,, \\ \tilde{y}_2 &= x_2 b_4^{-1/5}\,, \\ \tilde{y}_3 &= x_3 b_2^{-1/5}\,. \end{aligned} \tag{3.10}$$

With this substitution the equation (3.9) yields a curve of degree five, with two complex deformation parameters,

$$P(y_1, y_2, y_3) = x_1^5 + x_2^5 + x_3^5 + \zeta_1^{1/5} x_1 x_2 x_3^3 + \zeta_2^{1/5} x_1^2 x_2^2 x_3 \,, \tag{3.11}$$

where

$$\zeta_1 = \frac{b_0^5}{b_1 b_4 b_2^3} \qquad\qquad \zeta_2 = \frac{b_3^5}{b_1^2 b_4^2 b_2} \,. \tag{3.12}$$

We also define $z_1 = 1/\zeta_1$ and $z_2 = 1/\zeta_2$. Let us note that $(\zeta_1, \zeta_2) \longrightarrow (0,0)$ yield the orbifold limit in the mirror, while the LCSL is at the small values of z_1, z_2.

Since the large Kähler structure limit of $\mathbf{X}$ corresponds to the LCSL of the mirror $\check{\mathbf{X}}$, the B-type branes in the large volume limit correspond to A-type branes in the LCSL in the mirror. The coordinate chart suitable to the LCSL in $\check{\mathbf{X}}$ is given by combinations of the parameters b_i, with their indices in the combinations determined by the relations which correspond to the Gröbner basis of the toric ideal, as mentioned above. Thus, the good coordinates in the LCSL are

$$u = -\frac{b_0 b_1 b_4}{b_3^3} \qquad v = \frac{b_2 b_3}{b_0^2} \,, \tag{3.13}$$

corresponding to the relations $\ell^{(I)}$ in (3.5). These are related to the z_1, z_2 defined above by

$$z_1 = u v^3 \qquad z_2 = u^2 v \,. \tag{3.14}$$

The variables z_1 and z_2, as seen from (3.12), correspond to the relations

$$\begin{aligned} L^{(1)} &= (\, -5 \quad 1 \quad 3 \quad 0 \quad 1\,) \\ L^{(2)} &= (\, 0 \quad 2 \quad 1 \quad -5 \quad 2\,) \end{aligned} \tag{3.15}$$

in the same way u,v correspond to (3.5). The corresponding toric ideal is not Gröbner.

The discriminant locus of the curve (3.11) has three distinct components. The LCSL, given by $(u, v) = (0, 0)$, the orbifold point, given by $(\zeta_1, \zeta_2) = (0, 0)$ and finally, a curve, namely, the principal component, given by

$$(1 - 4v)^2 + 3125 u^2 v^3 - u(27 - 225v + 500v^2) = 0 \,. \tag{3.16}$$

3.4 The GKZ system

According to the local mirror conjecture the volume of the holomorphic cycles are related to the periods of a meromorphic one-form on the mirror geometry, the Riemann surface. In general it is not possible to integrate explicitly the one-form on the cycles of the Riemann surface. For a toric description there is a technical simplification due to the fact that all the periods satisfy a set of differential equations which one can directly obtain from the toric data. They are in general called $\mathcal{A}$-hypergeometric system and we are going construct the system for the present case.

Considering the affine space $\mathbb{C}^5$ with coordinates $b = \{b_0, \ldots, b_4\}$, for each $\ell \in \mathcal{L}_{\mathcal{A}}$, we define the homogeneous differential operator [22, 23, 24]

$$\mathcal{D}_\ell = \prod_{\ell_i > 0} \left[\frac{\partial}{\partial b_i}\right]^{\ell_i} - \prod_{\ell_i < 0} \left[\frac{\partial}{\partial b_i}\right]^{-\ell_i} . \tag{3.17}$$

The set of solutions $\Phi(b)$ to the system of equations

$$\mathcal{D}_{\ell^{(I)}} \Phi(b) = 0 \,, \tag{3.18}$$

corresponding to a basis of $\mathcal{L}_{\mathcal{A}}$ furnish the periods in some region of the moduli space of complex structure deformations.

For the toric data $\mathcal{A}$, the periods in LCSL are given by the kernels of the GKZ system of partial differential operators, written in terms of the variables u,v. Since $\mathsf{vol}(\mathcal{A}) = 5$, as can be verified from Figure 3, there are five linearly independent solutions to the GKZ system. These operators corresponds to the basis of the lattice of relations are

$$\begin{aligned} \mathcal{D}_1 &= (\Theta_u - 2\Theta_v)\Theta_u^2 - u(\Theta_v - 3\Theta_u)(\Theta_v - 3\Theta_u - 1)(\Theta_v - 3\Theta_u - 2) \,, \\ \mathcal{D}_2 &= (\Theta_v - 3\Theta_u)\Theta_v - v(\Theta_u - 2\Theta_v)(\Theta_u - 2\Theta_v - 1) \,, \end{aligned} \tag{3.19}$$

where $\Theta_x = x\frac{d}{dx}$ denotes the logarithmic derivative with respect to x.

Let us also record the GKZ operators in terms of the variable z_1,z_2.

$$\begin{aligned} \overset{\vee}{\mathcal{D}}_1 &= (2\Theta_2 + \Theta_1)^2(3\Theta_1 + \Theta_2)(2\Theta_2 + \Theta_1 - 1)^2 + \\ &\quad + z_2\Theta_2 \left(\Theta_2 + \frac{1}{5}\right)\left(\Theta_2 + \frac{2}{5}\right)\left(\Theta_2 + \frac{3}{5}\right)\left(\Theta_2 + \frac{4}{5}\right) , \end{aligned} \tag{3.20}$$

$$\begin{aligned} \overset{\vee}{\mathcal{D}}_2 &= (2\Theta_2 + \Theta_1)^2(3\Theta_1 + \Theta_2)(3\Theta_1 + \Theta_2 - 1)(3\Theta_1 + \Theta_2 - 2) + \\ &\quad + z_1\Theta_1 \left(\Theta_1 + \frac{1}{5}\right)\left(\Theta_1 + \frac{2}{5}\right)\left(\Theta_1 + \frac{3}{5}\right)\left(\Theta_1 + \frac{4}{5}\right) . \end{aligned} \tag{3.21}$$

where $\Theta_i = z_i\frac{d}{dz_i}$ for $i = 1, 2$. Both the equations being of fifth order, this system also has five independent solutions.

4. In the large complex structure limit

In this section we shall obtain all the periods near the LCSL. The periods on the cycles in **X** for the case at hand are in one-to-one correspondence with the solutions to the GKZ-system. The n-cycle periods can be identified, up to subleading terms, with the solutions to the GKZ-system whose leading terms are n^{th} powers (or their combinations) of the logarithm of the moduli space variables. A complete identification, however, is not straightforward. One way to fix that is through the monodromies

JHEP07(2001)007

around all the singularities of the moduli space. In the present example, however, the principal component of discriminant locus is a complicated curve. Hence we shall not calculate the monodromy around that. We shall make a few simplifying assumptions to obtain all the solutions.

We start by finding out the solutions to the GKZ system using Frobenius' method around the point $(u, v) = (0, 0)$. These correspond to the periods of cycles of the exceptional divisor of the blow up in the LCSL. In terms of the differential equations, the LCSL, $(u, v) = (0, 0)$, is characterised by the maximal degeneration of the solutions to the indicial equations associated to $\mathcal{D}_1$ and $\mathcal{D}_2$, which correspond to the Mori basis of the relations, ℓ^I [43]. More specifically, all the indices are zero. Therefore, in this limit, the equations have a single power series solution, two solutions containing logarithms of u and v, as well as two solutions containing squares of the logarithms.

The power series solution, also called the fundamental period, can be obtained from the relation $\ell^{(I)}$. Let us write it as

$$\varpi_0 = \sum_{m,n=0}^{\infty} a(m,n) u^m v^n \,. \tag{4.1}$$

The coefficients $a(m, n)$ of the series satisfy the recursion relations

$$\begin{aligned} \frac{a(m+1,n)}{a(m,n)} &= \frac{(n-3m)(n-3m-1)(n-3m-2)}{(m-2n+1)(m+1)}\,, \\ \frac{a(m,n+1)}{a(m,n)} &= \frac{(m-2n)(m-2n-1)}{(n-3m+1)(n+1)}\,, \end{aligned} \tag{4.2}$$

derived from (3.19). From these relations we can find out the associated radii of convergence of the series as the upper bounds on u, v. The domain of convergence is bounded by $|u| < u_0$ and $|v| < v_0$. The associated radii of convergence u_0 and v_0 trace out a curve in the u–v plane. The curve is parametrised by x and y, defined through

$$u_0 = \frac{x^2(x-2y)}{(y-3x)^3}\,, \qquad v_0 = \frac{y(y-3x)}{(x-2y)^2} \tag{4.3}$$

and is given by the following equation in terms of u_0 and v_0

$$(1-4v_0)^2 + 3125u_0^2v_0^3 - u_0(27 - 225v_0 + 500v_0^2) = 0\,. \tag{4.4}$$

Let us note that this is the same as the principal component of the discriminant locus (3.16) with u and v replaced by u_0 and v_0, respectively. From this equation, we also find that $\mathsf{max}[u_0] = \frac{1}{27}$ and $\mathsf{max}[v_0] = \frac{1}{4}$. The coefficient $a(m, n)$ of ϖ_0 in this domain, which solves the recursion relations and conforms to the toric data, is given by

$$a(m,n) = \frac{1}{\Gamma(1+m)^2\,\Gamma(1+n)\,\Gamma(1+m-2n)\,\Gamma(1-3m+n)}\,. \tag{4.5}$$

Let us note that with this coefficient, the only contribution to $\varpi_0(u,v)$ is from the term $m=n=0$, since all the other terms have poles of the Gamma functions in the denominator of the series. Hence $\varpi_0 = 1$.

The other solutions to the GKZ system can be obtained from ϖ_0 by taking indicial derivatives. The two solutions linear in the logarithms of u and v are given by

$$\begin{aligned}
\varpi_1 &= \lim_{\rho\longrightarrow 0} \frac{1}{2\pi i}\partial_\rho \left[\sum_{m,n=0}^{\infty} a(m+\rho,n) u^{m+\rho} v^n\right], \\
\varpi_2 &= \lim_{\rho\longrightarrow 0} \frac{1}{2\pi i}\partial_\rho \left[\sum_{m,n=0}^{\infty} a(m,n+\rho) u^{m} v^{n+\rho}\right].
\end{aligned} \tag{4.6}$$

The two solutions quadratic in the logarithms, on the other hand, are obtained by taking linear combinations of the second derivatives with respect to the indices, with coefficients determined by the Yukawa couplings as [41, 42]

$$\begin{aligned}
\varpi_3 &= \lim_{\rho_i\longrightarrow 0} \frac{1}{2!}\frac{1}{(2i\pi)^2}\sum_{j,k=1}^{2} c_{2jk}\partial_{\rho_j}\partial_{\rho_k} \left[\sum_{m,n=0}^{\infty} a(m+\rho_1,n+\rho_2) u^{m+\rho_j} v^{n+\rho_k}\right], \\
\varpi_4 &= \lim_{\rho_i\longrightarrow 0} \frac{1}{2!}\frac{1}{(2i\pi)^2}\sum_{j,k=1}^{2} c_{1jk}\partial_{\rho_j}\partial_{\rho_k} \left[\sum_{m,n=0}^{\infty} a(m+\rho_1,n+\rho_2) u^{m+\rho_j} v^{n+\rho_k}\right].
\end{aligned} \tag{4.7}$$

Using the definitions (4.6) and (4.7), let us now write down the explicit expressions of the solutions. Unlike ϖ_0, in all these four solutions the series receive contributions from non-vanishing m and n. We have

$$\begin{aligned}
\varpi_1(u,v) = {} & \frac{\varpi_0}{2\pi i}\log u + \frac{3}{2\pi i}\sum_{(n,r)'} \frac{\Gamma(5n+3r)}{\Gamma(1+r)\,\Gamma(1+n)\,\Gamma(1+2n+r)^2}(-u^2v)^n(-u)^r - \\
& -\frac{1}{2\pi i}\sum_{(m,r)'}\frac{\Gamma(5m+2r)}{\Gamma(1+r)\,\Gamma(1+3m+r)\,\Gamma(1+m)^2}(-uv^3)^m v^r\,,
\end{aligned} \tag{4.8}$$

where we used an abbreviation: $(m,n)' \equiv (m,n) \in \{\mathbb{Z}_+^2 \setminus (0,0)\}$. The other period linear in the logarithm is

$$\begin{aligned}
\varpi_2(u,v) = {} & \frac{\varpi_0}{2\pi i}\log v - \frac{1}{2\pi i}\sum_{(n,r)'} \frac{\Gamma(5n+3r)}{\Gamma(1+r)\,\Gamma(1+n)\,\Gamma(1+2n+r)^2}(-u^2v)^n(-u)^r + \\
& +\frac{2}{2\pi i}\sum_{(m,r)'}\frac{\Gamma(5m+2r)}{\Gamma(1+r)\,\Gamma(1+3m+r)\,\Gamma(1+m)^2}(-uv^3)^m v^r.
\end{aligned} \tag{4.9}$$

While the solutions are written in the chart (u,v), in the neighbourhood of $(0,0)$, each solution splits into terms whose "natural" variables are (z_1,v) and (u,z_2). We may, thus, take linear combinations of ϖ_1 and ϖ_2, which are expressed in these mixed set of variables. The combination

$$\begin{aligned}\Pi_1 &= \varpi_1 + 3\varpi_2 \\ &= \frac{1}{2\pi i}\log z_1 + \frac{5}{2\pi i}\sum_{(m,r)'}\frac{\Gamma\left(5m+2r\right)}{\Gamma\left(1+r\right)\Gamma\left(1+3m+r\right)\Gamma\left(1+m\right)^2}(-z_1)^m v^r\,, \end{aligned} \tag{4.10}$$

is expressed in terms of z_1 and v, while the combination

$$\begin{aligned}\Pi_2 &= \varpi_2 + 2\varpi_1 \\ &= \frac{1}{2\pi i}\log z_2 + \frac{5}{2\pi i}\sum_{(n,r)'}\frac{\Gamma\left(5n+3r\right)}{\Gamma\left(1+r\right)\Gamma\left(1+2n+r\right)^2\Gamma\left(1+n\right)}(-z_2)^n(-u)^r \end{aligned} \tag{4.11}$$

is expressed in terms of u and z_2. The solutions quadratic in the logarithms can also be found similarly. One of them, ϖ_3 takes the following form,

$$\begin{aligned}\varpi_3 = &-\frac{1}{10}\frac{1}{(2\pi i)^2}\left[2(\varpi_1+3\varpi_2)\log z_1 - \varpi_0(\log z_1)^2 - 36\Psi'(1)\right] - \\ &-\frac{1}{(2\pi i)^2}\sum_{(m,r)'}\frac{5\Psi(5m+2r)-2\Psi(1+m)-3\Psi(1+3m+r)}{\Gamma\left(1+r\right)\Gamma\left(1+m\right)^2\Gamma\left(1+3m+r\right)}\times \\ &\times\Gamma\left(5m+2r\right)(-z_1)^m v^r\,. \end{aligned} \tag{4.12}$$

The two solutions linear in logarithms represent the two semi-periods which are identified with the Kähler moduli using local mirror symmetry [41, 42, 38]. According to the principle of local mirror symmetry, the flat coordinates t_1 and t_2 in the LCSL have the asymptotic behaviours [44]

$$t_1 = \frac{1}{2\pi i}\log(-u) + \mathcal{O}(u,v)\,, \qquad t_2 = \frac{1}{2\pi i}\log(v) + \mathcal{O}(u,v)\,, \tag{4.13}$$

which imply the following identifications

$$t_1 = \varpi_1 + \frac{1}{2}\,, \qquad t_2 = \varpi_2 \tag{4.14}$$

between the flat coordinates and the periods.

The other two semi-periods dual to these are not exactly equal to the ϖ_3, ϖ_4. There is an ambiguity of adding linear combinations of ϖ_1 and ϖ_2 to these. In order to fix the periods dual to t_1 and t_2 we demand that in the LCSL they asymptote to

$$t_{d1} = \frac{1}{2}\sum_{i,j}c_{1ij}t_i t_j + m_1 + \mathcal{O}(u,v)\,, \tag{4.15}$$

$$t_{d2} = \frac{1}{2}\sum_{i,j}c_{2ij}t_i t_j + m_2 + \mathcal{O}(u,v)\,, \tag{4.16}$$

where m_1 and m_2 are two constants, interpreted as the mass of a D4-brane wrapped on the cycle. We shall denote the solutions that asymptote to t_{d1} and t_{d2} as Π_3 and

Π_4, respectively. This fixes the integral symplectic transformation that shifts the t_{di} by the t_i. The constants may be fixed by demanding that they should vanish on the principal component of the discriminant locus. We shall fix these later from the equality of the masses of the fractional branes. While discussing the identification of the brane configurations we shall find this to be the natural choice.

With the above choice of asymptotics we can now write down the solutions quadratic in logarithms as follows. One of them, Π_3, is naturally expressed in terms of the mixed variables as

$$\begin{aligned}\Pi_3(z_1, z_2) = &-\frac{1}{10}\frac{1}{2\pi i}\left[\log^2(-z_1) + \frac{10}{2\pi i}\log(-z_1) \times \right.\\ &\qquad\qquad \left. \times \sum_{(m,r)'} \frac{\Gamma\left(5m+2r\right)}{\Gamma\left(1+r\right)\Gamma\left(1+3m+r\right)\Gamma\left(1+m\right)^2}(-z_1)^m v^r\right] - \\ &-\frac{1}{(2\pi i)^2}\sum_{(m,r)'}\frac{5\Psi(5m+2r) - 2\Psi(1+m) - 3\Psi(1+3m+r)}{\Gamma\left(1+r\right)\Gamma\left(1+m\right)^2\Gamma\left(1+3m+r\right)} \times \\ &\times\Gamma\left(5m+2r\right)(-z_1)^m v^r + \frac{1}{10}\frac{18\Psi'(1)}{(2\pi i)^2}\,. \end{aligned} \tag{4.17}$$

The other solution quadratic in the logarithms, Π_4, can be succinctly expressed through the combination $\Pi_3 - 3\Pi_4$ as

$$\Pi_3 - 3\Pi_4 = \frac{1}{2}\frac{1}{(2\pi i)^2}\log^2(-u) + \frac{1}{2\pi i}\log(-u)\varpi_1 - \frac{12\psi'(1)}{2(2\pi i)^2}\,. \tag{4.18}$$

It may be noted that Π_3 and Π_4 are linear combinations of ϖ_4 and ϖ_3, respectively, with the other two semi-periods.

The monodromy around the point $(u, v) = (0, 0)$ can be obtained from the above formulas by expressing the changes of the solutions while going around the loops $(u, v) \longrightarrow (e^{2i\pi}u, v)$ and $(u, v) \longrightarrow (u, e^{2i\pi}v)$ in terms of the original solutions.

We shall find shortly that expressions of the solutions in terms of the mixed variables are easier for analytic continuation to the orbifold region. The second solution quadratic in logarithms, namely Π_4, does not admit a similar form, thus rendering its continuation to the orbifold region difficult.

5. At the orbifold point

The semi-periods obtained in the last section correspond to branes wrapped on the cycles at large volume. The fractional branes pertain to the region of the moduli space where the exceptional cycles shrink, giving way to the orbifold singularity. Thus, in order to compare the wrapped branes and the fractional branes we have to relate the periods in these two regions. One way of doing this is by analytic continuation of the large volume periods to the orbifold region. In the local mirror

JHEP07(2001)007

description (3.11), the orbifold point is located at $(\zeta_1, \zeta_2) \longrightarrow (0,0)$, which, in turn, corresponds to the point $(u,v) \longrightarrow (\infty, \infty)$. In this section we shall first perform the analytic continuation of the periods from the LCSL to the orbifold point. We shall then obtain the monodromy transformation of the periods and thence obtain the bound states of D4-, D2- and D0-branes which are identified with the fractional branes as the exceptional divisor shrinks.

5.1 Analytic continuation of periods

Analytic continuation of the periods is performed with the help of Mellin-Barnes-type integral representations of the periods obtained in the last section [45, 46, 47, 15]. Let us begin with Π_1. We write it as a contour integral as

$$\Pi_1 = \frac{1}{(2\pi i)^2} \sum_{r=0}^{\infty} \frac{v^r}{\Gamma(1+r)} \oint \frac{\Gamma(1+s)^2 \, \Gamma(-s)^2}{\Gamma(1+s)^2 \, \Gamma(1+3s+r) \, \Gamma(1-5s-2r)} z_1^s \,, \tag{5.1}$$

where the contour of integration encloses the poles of $\Gamma(-s)$ at the positive integral values of s. For analytic continuation, we first rewrite this as

$$\Pi_1 = \sum_{r=0}^{\infty} \frac{v^r}{\Gamma(1+r)} \oint \frac{\Gamma(5s+2r) \, \Gamma(-s)}{\Gamma(1+s) \, \Gamma(1+3s+r)} \phi(s) z_1^s \,, \tag{5.2}$$

where we defined

$$\begin{aligned} \phi(x) &= \frac{\sin 5\pi x}{\sin \pi x} \\ &= 5 - 20 \sin^2 x + 16 \sin^4 x \,. \end{aligned} \tag{5.3}$$

It can be checked that the last two terms in ϕ do not contribute to the integral in (5.2). Thus, we can set ϕ to 5 in (5.2) and obtain,

$$\Pi_1 = 5 \sum_{r=0}^{\infty} \frac{v^r}{\Gamma(1+r)} \oint \frac{\Gamma(5s+2r) \, \Gamma(-s)}{\Gamma(1+s) \, \Gamma(1+3s+r)} z_1^s \,. \tag{5.4}$$

The continued expressions are obtained on evaluating the integral after deforming the contour of integration to enclose poles on the negative real-axis in the s-plane. This yields

$$\Pi_1(\zeta_1, \zeta_2) = -\frac{5}{2\pi i} \sum_{(m,r)'} \frac{\Gamma\left(\frac{2r+m}{5}\right) {\zeta_1}^{\frac{m}{5}} {\zeta_2}^{\frac{r}{5}}}{\Gamma(1+r) \, \Gamma(1+m) \, \Gamma\left(1 - \frac{2r+m}{5}\right) \Gamma\left(1 - \frac{3m+r}{5}\right)} (-1)^m \,. \tag{5.5}$$

The solution Π_3 quadratic in logarithms can be continued in a similar fashion by first writing it as

$$\Pi_3 = -\frac{1}{5} \frac{1}{(2\pi i)^3} \sum_{r=0}^{\infty} \frac{v^r}{\Gamma(1+r)} \oint \frac{\Gamma(1+s)^3 \, \Gamma(-s)^3}{\Gamma(1+s)^2 \, \Gamma(1+3s+r) \, \Gamma(1-5s-2r)} (-z_1)^s + m_2 \,, \tag{5.6}$$

where again the contour encloses poles on the positive real s-axis. In continuing the solutions quadratic in the logarithms, the $\sin^2 x$ term in ϕ contributes to the integral by a constant. The contribution to Π_3 is absorbed in the constant m_2, which will be left arbitrary for the moment. We shall fix it later in this section. There is again a similar expansion as in (5.3), and the constant coming from the second term while last term will not contribute. Then using identities of Gamma function as before and deforming the contour we continue it to

$$\Pi_3(\zeta_1,\zeta_2) = \frac{1}{(2\pi i)^2}\sum_{(m,r)'}\frac{\Gamma\left(\frac{2r+m}{5}\right)^2 {\zeta_1}^{m/5}{\zeta_2}^{r/5}}{\Gamma\left(1+r\right)\Gamma\left(1+m\right)\Gamma\left(1-\frac{3m+r}{5}\right)}(-1)^{\frac{4m-2r}{5}} + m_2\,. \tag{5.7}$$

A similar consideration applies for Π_2. Its continuation leads to the following expression.

$$\Pi_2(\zeta_1,\zeta_2) = -\frac{5}{2\pi i}\sum_{(m,r)'}\frac{\Gamma\left(\frac{3r+n}{5}\right){\zeta_1}^{r/5}{\zeta_2}^{n/5}}{\Gamma\left(1+r\right)\Gamma\left(1+n\right)\Gamma\left(1-\frac{2n+r}{5}\right)^2}(-1)^r(-1)^n\,. \tag{5.8}$$

From the expressions of Π_1 and Π_2 we can obtain the continuations of the semi-periods ϖ_1 and ϖ_2 by inverting the relations (4.10) and (4.11).

The other semi-period Π_4 does not yield to such simple manoevres. A linear combination of the two semi-periods can also be written as a simple contour integral as

$$\Pi_3 - 3\Pi_4 = -\frac{1}{(2\pi i)^4}\oint ds\oint dt\frac{\Gamma\left(1+s\right)\Gamma\left(-s\right)^3\Gamma\left(-t\right)}{\Gamma\left(1+s-2t\right)\Gamma\left(1-3s+t\right)}(-u)^s v^t + m_2 - 3m_1\,, \tag{5.9}$$

but continuation to the orbifold is rather cumbersome. The form of Π_4, however, can be obtained by demanding special properties of the monodromy transformation, as we shall discuss shortly. Here we quote the result thus obtained for completeness:

$$\Pi_4(\zeta_1,\zeta_2) = \frac{1}{(2\pi i)^2}\sum_{(m,r)'}\frac{\Gamma\left(\frac{2r+m}{5}\right)^2\Gamma\left(1-\frac{3m+r}{5}\right){\zeta_1}^{m/5}{\zeta_2}^{r/5}}{\Gamma\left(1+r\right)\Gamma\left(1+m\right)}(-1)^{\frac{2r+m}{5}} + m_1\,, \tag{5.10}$$

where once again, m_1 is an arbitrary constant to be fixed.

5.2 The monodromy

In order to identify the periods obtained above with the fractional brane states of the CFT, we now have to study the behavior of the periods under monodromy transformations around the orbifold point. The basis of periods that correspond to the boundary states, by virtue of forming an orbit of the monodromy matrix, can be obtained by taking the solutions around the orbifold point in a loop. Let us define ${\zeta'}_1$ and ${\zeta'}_2$, such that $\zeta_1'^5 = \zeta_1$ and $\zeta_2'^5 = \zeta_2$. The monodromy transformation with respect to these coordinates is

$$\mathfrak{m}:\ (\zeta_1',\zeta_2') \longrightarrow (\omega\zeta_1',\omega^2\zeta_2')\,. \tag{5.11}$$

Monodromy of the solutions around the orbifold point is given by the matrix representation of $\mathfrak{m}$ in the basis of solutions obtained above.

The monodromy matrix is obtained by expressing the transformation of the periods under (5.11) in terms of the original periods. We first note that, using the expressions derived above, under the monodromy transformation $\mathfrak{m}$, $\Pi_3 \longrightarrow \Pi_3 - \frac{1}{5}\Pi_1$. That the 4-cycle under monodromy goes over to a sum of the 2- and the 4-cycles has occurred in other similar examples [3, 4]. We fix the expression of Π_4, whose analytic continuation has not been performed in the previous subsection, by demanding that this 4-cycle volume under monodromy goes over to a combination of Π_4 and Π_2 only. This seems to be a general feature for all $\mathbb{Z}_N$ orbifolds. This fixes Π_4. We rewrite the periods at the orbifold point in a convenient form as

$$\begin{aligned}
\varpi_0 &= 1\,, \\
\Pi_1 &= \sum_{(m,n)'} (\omega^n - \omega^{4m+4n} - \omega^{2m} + \omega^{m+3n})\Pi(m,n)\,, \\
\Pi_2 &= \sum_{(m,n)'} (\omega^{4m+4n} - 2\omega^{3m+2n} + \omega^{2m})\Pi(m,n)\,, \\
\Pi_3 &= -\frac{1}{5}\sum_{(m,n)'} (\omega^{4m+4n} - \omega^{m+3n})\Pi(m,n) + m_2\,, \\
\Pi_4 &= -\frac{1}{5}\sum_{(m,n)'} (\omega^{3m+2n} - \omega^{2m})\Pi(m,n) + m_1\,,
\end{aligned} \tag{5.12}$$

where we have defined

$$\Pi(m,n) = -\frac{1}{(2\pi i)^3}\frac{\Gamma\left(\frac{m+2n}{5}\right)^2\Gamma\left(\frac{3m+n}{5}\right)}{\Gamma\left(1+m\right)\Gamma\left(1+n\right)}e^{\frac{i\pi}{5}(n-m)}\zeta_1'^m\zeta_2'^n\,. \tag{5.13}$$

The monodromy matrix in the basis of periods $(\varpi_0, \Pi_1, \Pi_2, \Pi_4, \Pi_3)$ is obtained from these expressions by applying $\mathfrak{m}$ on them and is given by

$$\mathbf{m} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5(m_1 - 2m_2) & -1 & 1 & -5 & 10 \\ 5(m_2 - 3m_1) & 1 & -2 & 15 & -5 \\ 0 & 0 & -1/5 & 1 & 0 \\ 0 & -1/5 & 0 & 0 & 1 \end{pmatrix}. \tag{5.14}$$

However, for the identification of fractional branes with the wrapped branes, we need to find out the branes wrapped on the cycles in the LCSL, forming an orbit of the monodromy matrix. Thus, we need the monodromy matrix in the basis of periods in the LCSL. In the LCSL basis $\varpi = (\varpi_0, \varpi_1, \varpi_2, \Pi_4, \Pi_3)$ of periods, the monodromy matrix becomes,[1]

$$\mathcal{M}(\varpi^{\Box}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5(m_2 - 2m_1) & -2 & 1 & 10 & -5 \\ 5(m_1 - m_2) & 1 & -1 & -5 & 5 \\ 0 & -2/5 & -1/5 & 1 & 0 \\ 0 & -1/5 & -3/5 & 0 & 1 \end{pmatrix}. \tag{5.15}$$

Due to the $\mathbb{Z}_5$ symmetry at the orbifold point, the monodromy matrix satisfies $\mathcal{M}^5 = \mathbb{I}_5$. The canonical symplectic basis of periods in the LCSL admits a pairing through the symplectic intersection matrix

$$\mathcal{I} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.16}$$

The intersection pairing between the combinations of the periods in the LCSL, forming an orbit of $\mathcal{M}$ yields the adjacency matrix of the quiver diagram of $\mathbb{Z}_5$, which we shall now discuss.

5.3 Charges and brane configurations

Having obtained the expressions of the periods (5.12) at the orbifold point, the monodromy $\mathcal{M}$ and their connection to the periods and monodromy in the LCSL, we can now compare the D-brane configurations in these two regimes. As mentioned before, at the orbifold point the basic objects are the fractional branes which are to be identified as bound states of wrapped branes in the large volume region. Due to the quantum $\mathbb{Z}_5$ symmetry at the orbifold point, the fractional branes form an orbit of the monodromy. Thus, on identifying a single fractional brane a set of five configurations can be generated by successive right-action of the monodromy matrix on the charges of original wrapped brane.

Let us start from a D4-brane wrapped on the four-cycles, ϖ_3 and ϖ_4, such that the charge in the basis ϖ of periods is $\mathcal{Q}_1 = (0, 0, 0, -1, 2)$. This state has mass $2m_1 - m_2$. Under the action of $\mathcal{M}(\varpi)$, this gives rise to the charges of four other states as $\mathcal{Q}_{i+1} = \mathcal{Q}_1 \mathcal{M}^i$, $i = 1, 2, 3, 4$. The five states thus obtained are bound states of one wrapped D0-brane and two D2- and two D4-branes, as can be identified from ϖ. The states are tabulated in table 2. Each of these states has a mass $2m_1 - m_2$, and since there are five states in the orbit, by virtue of $\mathcal{M}^5 = \mathbb{I}_5$, we conclude that $2m_1 - m_2 = 1/5$, the mass of each fractional brane in a unit in which the D0-brane

[1] The fractions appearing in the monodromy matrix are due to the choice of the normalisation $1/5$ in the solutions quadratic in logarithms. These were chosen as unity in [3, 4].

CHARGE	D0	$\mathrm{D2}_1$	$\mathrm{D2}_2$	$\mathrm{D4}_1$	$\mathrm{D4}_2$
$\mathcal{Q}_1$	0	0	0	-1	2
$\mathcal{Q}_2$	0	0	-1	-1	2
$\mathcal{Q}_3$	$5(m_2 - m_1)$	-1	0	4	-3
$\mathcal{Q}_4$	$5m_1$	1	0	-6	2
$\mathcal{Q}_5$	$5(m_2 - m_1)$	0	1	4	-3

Table 2: Central charges corresponding to the bound states of branes.

has unit mass. A numerical evaluation of m_1 and m_2 at one point of the principal component of the discriminant locus (3.16) confirms this conclusion.

The pairing of the charges $\mathcal{Q}_i$, $i = 1, \ldots, 5$, using the symplectic matrix (5.16), reproduces the adjacency matrix $\mathcal{C}$ of the quiver diagram of $\mathbb{Z}_5$, i.e.

$$C_{ij} = \mathcal{Q}_i^{\mathrm{T}} \mathcal{I} \mathcal{Q}_j \,, \tag{5.17}$$

where a superscript $^{\mathrm{T}}$ signifies matrix-transpose. The same has also been obtained from the boundary states describing fractional branes through the Witten index in section 2.3. Therefore, these five bound states of D-branes given by the charges $\mathcal{Q}_i$ can be identified as the five boundary states obtained there. In other words, this establishes the connection between the irreducible representations of the group $\mathbb{Z}_5$ and the combinations of the 0-, 2- and 4-cycle periods of the exceptional divisors of the crepant resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_5$, thus furnishing a D-brane realisation of the McKay correspondence.

A further check on the choice of the initial charge $\mathcal{Q}_1$ is as follows. The charges of the fractional branes $\mathcal{Q}_i$ have been conjectured [7, 11, 14, 15] to be dual to the tautological line bundles $\mathcal{R}_i$ on the resolution with respect to the pairing

$$\int \mathrm{ch}\mathcal{R}_i \cdot \mathrm{td}\mathbf{X} \cdot \mathrm{ch}\mathcal{Q}_j = \delta ij \,, \tag{5.18}$$

where $\mathrm{td}X$ denotes the Todd class of X and $\mathrm{ch}\mathcal{E}$ denotes the Chern character of a bundle $\mathcal{E}$. The charge vector $\mathcal{Q}_1$ chosen above may be checked to be dual to the trivial line bundle with respect to this pairing.

A specification of brane configurations entails, in addition to the dimensions of branes, the topological numbers associated to the gauge theory supported on them. This information, referred to as the bundle data, is obtained by comparing two expressions of the central charges [4] of the supersymmetry algebra of the gauge theory, whose equality is necessitated by anomaly considerations [48, 49]. One of the expressions for the central charges involves the periods and the charges described above. For the case at hand, a generic bound state corresponds to

1. D4-branes wrapped on a surface in a homology class $(n_1^{[4]}, n_2^{[4]})$ of the resolution

2. D2-branes corresponding to vector bundles with first Chern classes $(n_1^{[2]}, n_2^{[2]})$

3. D0-branes corresponding to an instanton number $n^{[0]}$.

On writing the charges in the basis $(t_{d1}, t_{d2}, t_1, t_2, 1)$ of periods as $(n_1^{[4]}, n_2^{[4]}, n_1^{[2]}, n_2^{[2]}, n^{[0]})$, the central charge of this configuration assumes the form

$$\mathscr{Q} = \sum_{i=1}^{2} n_i^{[4]} \cdot t_{di} + \sum_{i=1}^{2} n_i^{[2]} \cdot t_i + n^{[0]} \,. \tag{5.19}$$

In large volume limit (or LCSL), on the other hand, the central charge of a D4-brane wrapped on a surface S with a vector bundle $\mathcal{E}$ can be obtained from considering cancellation of anomalies and is given by

$$\mathscr{Q} = \int_S e^{-\mathbf{K}} \mathrm{ch}\mathcal{E} \sqrt{\frac{\hat{A}(T_S)}{\hat{A}(N_S)}} \,, \tag{5.20}$$

where $\mathbf{K}$ is the Kähler form and $\hat{A}(T_S)$ and $\hat{A}(N_S)$ are the respective A-roof genus of the Tangent bundle and the Normal bundle of S embedded in the resolution. The bundle data for the fractional branes can be obtained by equating these two expressions of central charges.

In order to compare the two expressions (5.19) and (5.20) for the central charge, we expand [3, 4, 10, 9] the Kähler form $\mathbf{K}$ in terms of t_i as $\mathbf{K} = \mathbf{t_1} \cdot \mathfrak{d}_\mathbf{1} + \mathbf{t_2} \cdot \mathfrak{d}_\mathbf{2}$. Here we use the symbol $\mathfrak{d}_i$ also to denote the restriction of divisor classes $\mathfrak{d}_i$ to S. Moreover, since S is holomorphically embedded in the resolution $\mathbf{X}$ which has vanishing first Chern class, we have

$$\sqrt{\frac{\hat{A}(T_S)}{\hat{A}(N_S)}} = \chi(S)\omega_S \,, \tag{5.21}$$

where $\chi(S)$ is the Euler number of the surface S and ω_S satisfies $\int_S \omega_S = 1$. The expression (5.20) thus reduces to a quadratic in t_i as follows

$$\mathscr{Q} = \frac{1}{2} \sum_{i,j=1}^{2} \int_S (\mathfrak{d}_i \cdot \mathfrak{d}_j) t_i \cdot t_j - \sum_{i=1}^{2} \int_S \mathfrak{d}_i \cdot \mathrm{ch}_1 \mathcal{E} \cdot t_i - \frac{1}{24} \chi(S) + \int_S \mathrm{ch}_2 \mathcal{E} \,. \tag{5.22}$$

If S belongs to a homology class $(n_1^{[4]}, n_2^{[4]})$, then we have, from the computation of the intersection numbers in section 3.2, the following expressions:

$$\begin{aligned} \int_S \mathfrak{d}_1^2 &= \frac{1}{5}(2n_1^{[4]} + n_2^{[4]}) \,, \\ \int_S \mathfrak{d}_2^2 &= \frac{3}{5}(n_1^{[4]} + 3n_2^{[4]}) \,, \\ \int_S (\mathfrak{d}_2^2 - 3\mathfrak{d}_1 \cdot \mathfrak{d}_2) &= 0 \,. \end{aligned} \tag{5.23}$$

The first term in (5.22) can be written as a sum of t_{d1} and t_{d2} using the expressions (4.16) and the intersection numbers. To simplify the second term, we write $\mathrm{ch}_1\mathcal{E} = a_1{\cdot}\mathfrak{d}_1 + a_2{\cdot}\mathfrak{d}_2$. Substituting this, the second term of (5.22) becomes

$$\sum_{i,j,k} c_{ijk} a_j n_k^{[4]}{\cdot}t_i = \sum_i n_i^{[2]}{\cdot}t_i\,. \tag{5.24}$$

The last term is, $n^{[0]} = \int_S \mathrm{ch}_2\mathcal{E}$. Combining these simplified expressions, we obtain, from (5.19), the expression (5.19) for the central charge $\mathscr{Q}$. This way of identifying the central charges (5.19) and (5.20) leads to the explicit specification of the rank and Chern characters of the bundles supported on the exceptional divisors on which the BPS branes are wrapped.

6. Summary

To summarise, in this article we have obtained the geometric realisations of the boundary states corresponding to fractional D0-branes in $\mathcal{N} = 2$ CFT on the orbifold $\mathbb{C}^3/\mathbb{Z}_5$. The orbifold admits a crepant resolution. The Newton polygon underlying this variety is not reflexive. The exceptional divisor of the resolution consists in $\mathbb{P}^2$ and $\mathbb{F}_2$. By studying the periods of cycles of the resolution with the help of the GKZ equations of its local mirror in the LCSL, we obtained the B-type branes wrapped on the holomorphic cycles of the exceptional divisors. Analytic continuation of the periods to the orbifold point provides the bound states of D0-, D2- and D4-branes that form an orbit of the monodromy matrix at that point. Identifying these five states as the five fractional branes by matching the pairing matrix between these and the open-string Witten indices derived from the CFT yields the large-volume brane configurations that correspond to the fractional brane boundary states. Taking into account the RR-charges coupling to the branes, this furnishes further evidence to identification of branes as vector bundles on the cycles of the resolution.

An understanding of D-branes wrapped on non-trivial cycles clarifies the behaviour of gauge theories with low supersymmetry in different phases and extends our understanding of D-branes. The identification of branes with bundles on the cycles is important for obtaining a precise mathematical definition of these objects. This precision also entails studies on the stability of the bounds states, and hence of the associated bundles. This generalises the concept of μ-stability of vector bundles to Π-stability. We have not considered the questions of stability of the bound states in this article. Such a study requires explicit expressions of the periods near the singular loci. While possible in principle, this is technically cumbersome, as we have seen already in section 5. We postpone it to a future study.

It will be interesting to generalise these considerations to $G = \mathbb{Z}_N$ for arbitrary N and for non-abelian cases. Some mathematical results for these cases are already available [32, 31]. This will also bring out a relationship between the generalised hypergeometric system and the tautological bundles on the resolutions of the orbifolds, in view of more recent considerations that describes the fractional branes as duals of tautological bundles [7, 11, 14, 15].

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References

[1] M.R. Douglas, *Enhanced gauge symmetry in M(atrix) theory*, *J. High Energy Phys.* **07** (1997) 004 [hep-th/9612126].

[2] D.-E. Diaconescu, M.R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, *J. High Energy Phys.* **02** (1998) 013 [hep-th/9712230].

[3] D.-E. Diaconescu and J. Gomis, *Fractional branes and boundary states in orbifold theories*, *J. High Energy Phys.* **10** (2000) 001 [hep-th/9906242].

[4] K. Mohri, Y. Onjo and S.-K. Yang, *Closed sub-monodromy problems, local mirror symmetry and branes on orbifolds*, hep-th/0009072.

[5] M.R. Douglas, B. Fiol and C. Romelsberger, *The spectrum of BPS branes on a non-compact Calabi-Yau*, hep-th/0003263.

[6] M.R. Douglas, B. Fiol and C. Romelsberger, *Stability and BPS branes*, hep-th/0002037.

[7] D.-E. Diaconescu and M.R. Douglas, *D-branes on stringy Calabi-Yau manifolds*, hep-th/0006224.

[8] I. Brunner, M.R. Douglas, A.E. Lawrence and C. Romelsberger, *D-branes on the quintic*, *J. High Energy Phys.* **08** (2000) 015 [hep-th/9906200].

[9] P. Kaste, W. Lerche, C.A. Lutken and J. Walcher, *D-branes on K3-fibrations*, *Nucl. Phys.* **B 582** (2000) 203 [hep-th/9912147].

[10] E. Scheidegger, *D-branes on some one- and two-parameter Calabi-Yau hypersurfaces*, *J. High Energy Phys.* **04** (2000) 003 [hep-th/9912188].

[11] S. Govindarajan and T. Jayaraman, *D-branes, exceptional sheaves and quivers on Calabi-Yau manifolds: from Mukai to McKay*, *Nucl. Phys.* **B 600** (2001) 457 [hep-th/0010196].

[12] S. Govindarajan, T. Jayaraman and T. Sarkar, *On D-branes from gauged linear sigma models*, *Nucl. Phys.* **B 593** (2001) 155 [hep-th/0007075].

[13] S. Govindarajan and T. Jayaraman, *On the Landau-Ginzburg description of boundary CFTS and special lagrangian submanifolds*, *J. High Energy Phys.* **07** (2000) 016 [hep-th/0003242].

[14] A. Tomasiello, *D-branes on Calabi-Yau manifolds and helices*, *J. High Energy Phys.* **02** (2001) 008 [hep-th/0010217].

[15] P. Mayr, *Phases of supersymmetric D-branes on kaehler manifolds and the McKay correspondence*, *J. High Energy Phys.* **01** (2001) 018 [hep-th/0010223].

[16] M Douglas, *D-branes on Calabi-Yau manifolds*, math.AG/0009209.

[17] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186].

[18] M. Gutperle and Y. Satoh, *D-branes in Gepner models and supersymmetry*, *Nucl. Phys.* **B 543** (1999) 73 [hep-th/9808080].

[19] A. Sagnotti, *Surprises in open-string perturbation theory*, *Nucl. Phys.* **56B** *(Proc. Suppl.)* (1997) 332 [hep-th/9702093].

[20] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y.S. Stanev, *Comments on Gepner models and type I vacua in string theory*, *Phys. Lett.* **B 387** (1996) 743 [hep-th/9607229].

[21] A.C. Avram, E. Derrick and D. Jancic, *On semi-periods*, *Nucl. Phys.* **B 471** (1996) 293 [hep-th/9511152].

[22] I Gel'fand, A Zelevinkii and M Kapranov, *Hypergemetric functions and toral manifolds*, *Func. Anal. and Appl.*, **23** (1989) 94.

[23] A. Klemm, B.H. Lian, S.S. Roan and S.T. Yau, *A note on odes from mirror symmetry*, hep-th/9407192.

[24] M. Koseki and R. Kuriki, *Quantization of 2D abelian gauge theory without the kinetic term of gauge field as anomalous gauge theory*, hep-th/9511001.

[25] S Hosono, *GKZ systems, Gröbner fans and moduli spaces of Calabi-Yau hypersurfaces*, alg-geom/9707003.

[26] P Horja, *Hypergeometric functions and mirror symmetry in toric varieties*, math.AG/9912109.

[27] M.R. Douglas and G. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[28] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[29] F. Roose, *Boundary states and non-abelian orbifolds*, hep-th/0002129.

[30] T. Takayanagi, *String creation and monodromy from fractional D-branes on ALE spaces*, *J. High Energy Phys.* **02** (2000) 040 [hep-th/9912157].

[31] M Reid, *McKay correspondence*, alg-geom/9702016.

[32] Y Ito and M Reid, *The McKay correspondence for finite subgroups of* $\mathrm{SL}(3, \mathbb{C})$, alg-geom/9410010.

[33] A Craw and M Reid, *How to calculate A-Hilb* $\mathbb{C}^3$, math.AG/9909085.

[34] A Craw, *An explicit construction of the McKay correspondence for A-Hilb* $\mathbb{C}^3$, math.AG/0010053.

[35] W. Fulton, *Introduction to toric varieties*, Princeton University Press, Princeton 1993.

[36] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of* $4D$ $N = 2$ *gauge theories I*, *Adv. Theor. Math. Phys.* **1** (1998) 53 [hep-th/9706110].

[37] S. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, *Nucl. Phys.* **B 497** (1997) 173 [hep-th/9609239].

[38] T.M. Chiang, A. Klemm, S.T. Yau and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, *Adv. Theor. Math. Phys.* **3** (1999) 495 [hep-th/9903053].

[39] S. Hosono, *Local mirror symmetry and type IIA monodromy of Calabi-Yau manifolds*, *Adv. Theor. Math. Phys.* **4** (2001) 335 [hep-th/0007071].

[40] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.

[41] S. Hosono, A. Klemm, S. Theisen and S.T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, *Comm. Math. Phys.* **167** (1995) 301 [hep-th/9308122].

[42] S. Hosono, A. Klemm, S. Theisen and S.T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, *Nucl. Phys.* **B 433** (1995) 501 [hep-th/9406055].

[43] S Hosono, B Lian and S Yau, *Maximal degeneracy points of GKZ systems*, *J. Amer. Math. Soc.*, **10(2)** (1997) 427, alg-geom/9603014.

[44] P.S. Aspinwall, *Resolution of orbifold singularities in string theory*, hep-th/9403123.

[45] P Candelas, X de la Ossa, P Green and L Parkes, *A pair of Calabi Yau manifolds as an exactly solvable superconformal theory*, *Nucl. Phys.* **B 356** (1991) 21.

JHEP07(2001)007

[46] P. Candelas, X.D. L. Ossa, A. Font, S. Katz and D.R. Morrison, *Mirror symmetry for two parameter models I*, *Nucl. Phys.* **B 416** (1994) 481 [hep-th/9308083].

[47] P. Candelas, A. Font, S. Katz and D.R. Morrison, *Mirror symmetry for two parameter models II*, *Nucl. Phys.* **B 429** (1994) 626 [hep-th/9403187].

[48] Y.-K.E. Cheung and Z. Yin, *Anomalies, branes and currents*, *Nucl. Phys.* **B 517** (1998) 69 [hep-th/9710206].

[49] C.A. Scrucca and M. Serone, *Anomaly inflow and RR anomalous couplings*, hep-th/9911223.